J. Fluid Mech. (1980), *vol.* 98, *part.* 3, *pp.* 527–553.
Printed in Great Britain

Energy stability theory for free-surface problems: buoyancy–thermocapillary layers

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(Received 14 May 1979 and in revised form 8 October 1979)

Energy stability theory has been formulated for two-dimensional buoyancy–thermocapillary convection in a layer with a free surface. The theory yields a critical Rayleigh number R_E for which $R < R_E$ is a sufficient condition for stability of the layer. R_E emerges from the variational formulation as an eigenvalue of a *nonlinear system* of Euler–Lagrange equations. For the case of small capillary number (large mean surface tension) explicit values are obtained for R_E. The analogous linear-theory results for this case are obtained in terms of a critical Rayleigh number R_L. These are compared. It is found that the existence of the deformable interface can lead to a stabilization relative to the case of a planar interface. This result is explained in physical terms. The energy theory is then generalized to include general flow problems having three-dimensional disturbances, non-Newtonian bulk fluids and general interfacial mechanics such as surface viscosity and elasticity.

1. Introduction

Free-surface problems in continuum mechanics have a characteristic nonlinearity which results from the fact that the free-surface position is *a priori* unknown. Free-surface boundary conditions are applied on a surface whose location is part of the unknown solution. This characteristic nonlinearity persists even if the system involves only linear materials in the bulk and linear materials on the surface.

Stability theories of free-surface systems must confront such nonlinearity. These stability theories in addition must confront other behaviours characteristic of free-surface problems, among which are inherent non-uniqueness and trajectory splitting.

All of the above behaviours can be illustrated by means of a simple example shown in figure 1. Consider an infinite container of Newtonian liquid. The liquid possesses surface tension on its interface with the bounding gas; body forces are absent and the system is stationary. This static configuration is (locally) stable. Consider, however, a disturbance of the interface such that at time $t = 0$ the interface consists of a 'flat' portion nearly everywhere plus a tall, isolated, axisymmetric spike as shown in figure 1(*b*). It is *conceivable* that, as time increases, the surface tension will cause droplets to form owing to capillary instability (Rayleigh 1879). In this case at a time $t = t_1$, an incipient droplet will be present as shown in figure 1(*c*). At a larger time

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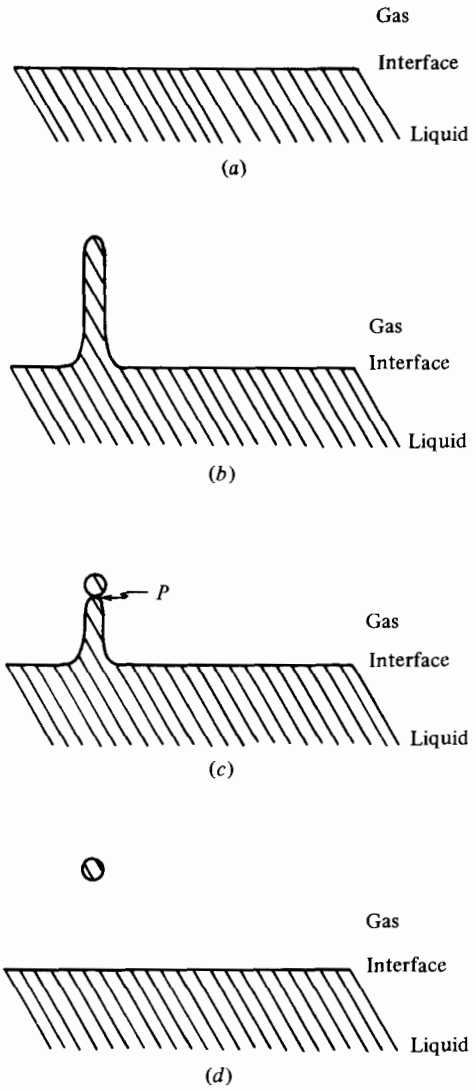

FIGURE 1. Sketch of the configurations of a free surface in a thought experiment. Body forces are absent and there is surface tension on the interface. (*a*) The undisturbed state at $t < 0$. (*b*) The initial configuration at $t = 0$ containing an axisymmetric spike. (*c*) An incipient droplet formed at $t = t_1 > 0$ by capillary instability where P denotes the one common point between the droplet and the spike. (*d*) The final state, $t \to \infty$, consisting of a single spherical droplet plus the infinite bath having a flat interface.

$t = t_2$, full spherical droplets, perhaps with smaller satellite droplets, will be formed while the remainder of the spike will have decayed into the bath through capillary forces. As $t \to \infty$ the system consisting of isolated, spherical drops plus the infinite bath with a planar interface will itself be a (locally) stable configuration. Clearly, there is infinite non-uniqueness in this system since any number of spherical droplets plus the bath comprises a (locally) stable configuration. Furthermore, at time $t = t_1$, there is a trajectory that is not well defined. The point labelled P in figure 1(*c*) will split

into two points, one associated with the droplet and one associated with the remaining spike. This splitting will generate a singular velocity field (Dussan V. & Davis 1974).

We wish to consider the formulation of a stability theory incorporating the effects of disturbances of finite amplitude and capable of giving stability limits for free-surface flows of quite general types. It is clear from the above illustration that such a theory must be *nonlinear* and will generally be a *conditional theory*, i.e. the stability limits are expected to be dependent on a norm (or amplitude) of the disturbance quantities. For example, the planar interface in the above illustration might be stable if certain tall, slender interfacial disturbances are excluded. This exclusion might take the form of a certain surface norm being sufficiently small.

Stability theories that incorporate the nonlinearities of the problem are either weakly nonlinear theories which pivot a perturbation procedure about a critical point of linear stability theory, or energy theories that use variational procedures. We concern ourselves here with the latter.

There has only been one paper aimed at the development of an energy stability theory for free-surface flows. Dussan V. (1975) formulated the appropriate energy theory for a general system whose interface has constant surface tension. She was able to relate the surface-stress boundary conditions to changes in surface energy and obtain sharp stability and instability results for the Rayleigh–Taylor problem. Her results have a range of conditional stability, expressed in terms of the L_2 norm of the surface displacement. Her formulation also provides an elegant link between the static-stability idea of physical chemistry and hydrodynamic theories applied to static systems. However, her formulation is not convenient for systems that are inherently dynamic, and does not make clear the ability to incorporate properties more exotic than constant surface tension (e.g. variable surface tension or surface viscosity).

The present work aims at presenting a general approach to the formulation of energy stability theory for free-surface flows. For simplicity, we examine a specific problem and formulate the theory for two-dimensional buoyancy–thermocapillary instability in a horizontal fluid layer (§ 2) and obtain a statement of the energy theory and the associated Euler–Lagrange equations in § 3. This theory supplies a critical value R_E^2 of the Rayleigh number R^2 for which $R < R_E$ gives a sufficient condition for stability. For this particular problem we explore in detail the small capillary-number behaviour (§ 4). In this limiting case we regain and (for two-dimensional disturbances) strengthen the results of Davis (1969), who presumed *a priori* that the free surface is planar. We obtain in § 5 some linear-theory results for the same problem in terms of a critical value R_L of R. In § 6 we compare R_E and R_L, thus giving a picture of the global-stability behaviour. When $R < R_E$, the static layer is stable and when $R > R_L$ it is unstable. When $R_E < R < R_L$, finite-amplitude subcritical instabilities may be present. Of particular interest is the illustration, through these perturbation results, of the conditional nature of the stability statements, and the connexion between the norm of the surface displacements and the physical variables of the problem. We then generalize (§ 7) the approach to general flow problems having three-dimensional disturbances, having general bulk constitutive behaviour, and general interfacial behaviour (surface excess mass, surface-tension gradients, surface viscosities, etc.).

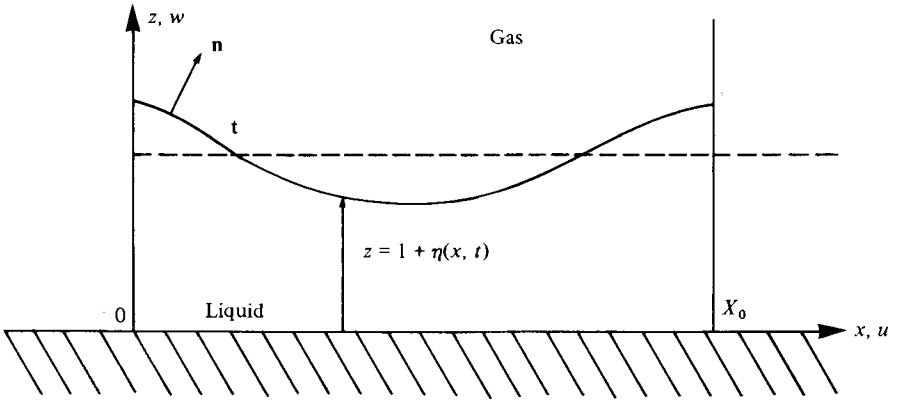

FIGURE 2. A sketch of the 'volume' of a single wavelength of the disturbed layer having unit mean depth, wavelength X_0 and height $z = 1+\eta(x, t)$. $\mathbf{n}$ is the unit outward normal to the disturbed interface.

2. Formulation: two-dimensional buoyancy–thermocapillary instability

(a) *The governing system*

We shall use the following notation (refer to figure 2): d is the mean distance between two infinite horizontal surfaces; the lower surface is a rigid plane at constant temperature, while the upper surface is a free surface on which the heat flux is prescribed. These surfaces bound an incompressible, Newtonian liquid of mean density ρ, viscosity μ and kinematic viscosity $\nu = \mu/\rho$. The acceleration of gravity is g; α and κ are respectively the coefficients of thermal expansion and thermal diffusivity of the liquid. The space above the free surface contains a passive gas have negligible density and viscosity. We associate with the free surface a surface tension which varies with temperature.

We use a system of Cartesian co-ordinates whose origin lies in the rigid plate and whose dimensionless vertical co-ordinate z and dimensionless horizontal co-ordinate x are scaled on d. We thus consider two-dimensional fields, but indicate the generalization to three dimensions in § 7. The velocity vector $\mathbf{v} = (u, w)$, the temperature θ, the time t, the pressure p, and the surface tension σ are referred to scales κ/d, ΔT, d^2/κ, $\mu\kappa/d^2$, σ_0, where $\Delta T(> 0)$ is the temperature excess at the bottom compared to the top and σ_0 is the mean surface tension on the free surface. We employ the Boussinesq approximation. Under the above assumptions, the following non-dimensional groups emerge:

$$R^2 = \alpha\Delta T g d^3/\kappa\nu \quad \text{Rayleigh number;} \tag{2.1a}$$

$$M = \sigma_1 \Delta T d/\mu\kappa \quad \text{Marangoni number;} \tag{2.1b}$$

$$P = \nu/\kappa \quad \text{Prandtl number;} \tag{2.1c}$$

$$C = \mu\kappa/\sigma_0 d \quad \text{capillary number;} \tag{2.1d}$$

$$G = \rho g d^2/\sigma_0 \quad \text{Bond number.} \tag{2.1e}$$

R^2 and M are dimensionless measures of the importance of the two instability mechanisms to which the system is susceptible. In the definition of M, σ_1 is the (constant) negative of the derivative of surface tension with respect to temperature. We therefore use a linear equation of state for surface tension. In particular, in non-dimensional form

$$\sigma(\theta) = 1 - MC\theta. \tag{2.2}$$

The capillary number C measures the degree of deformability of the free surface since $C \to 0$ implies that the mean surface tension is very large. This interpretation will be made more explicit in §§ 4 and 5 below, where we discuss solutions to the stability problem for small C.

In terms of the above parameters, the Boussinesq approximation requires that, if

$$\delta \equiv \alpha \Delta T, \quad \delta \to 0 \quad \text{for } R^2 \text{ fixed}.$$

Note that δ is not independent of the above groups but $\delta = R^2 C/G$.

We locate the liquid–gas interface $S(t)$ by writing

$$S(t)\colon z = 1 + \eta(x, t).$$

For a sufficiently smooth general two-dimensional deformation of the interface, the unit, outward normal vector $\mathbf{n}$ to S has the form

$$\mathbf{n} = (-\eta_x, 1)/N, \tag{2.3a}$$

where

$$N \equiv (1 + \eta_x^2)^{\frac{1}{2}}. \tag{2.3b}$$

We define the unit tangent vector $\mathbf{t}$ to S as follows:

$$\mathbf{t} = (1, \eta_x)/N. \tag{2.3c}$$

The stress balance at the interface may be written as a single vector equation:

$$\sigma_{ij} n_j = \frac{1}{C} K(\eta)\, \sigma(\theta)\, n_i - M \theta_{,k} t_k t_i \quad \text{on} \quad z = 1 + \eta. \tag{2.4}$$

Here σ_{ij} is the stress tensor of the liquid

$$\sigma_{ij} = -p\delta_{ij} + \epsilon_{ij}, \quad \epsilon_{ij} = v_{i,j} + v_{j,i}, \tag{2.5a,b}$$

and $K(\eta)$ is the curvature,

$$K(\eta) = \eta_{xx}/N^3. \tag{2.5c}$$

Commas denote spatial differentiation, δ_{ij} is the Kroneker delta and the summation convention is assumed.

The kinematic boundary condition takes the form

$$\eta_t = N v_i n_i \quad \text{on} \quad z = 1 + \eta. \tag{2.6}$$

For simplicity, we assume that the heat flux on S is prescribed,

$$\theta_{,i} n_i = -1 \quad \text{on} \quad z = 1 + \eta. \tag{2.7}$$

This is most appropriate for convection over a gas, and its adoption here has some implications with regard to the perturbation results of § 4, which we shall comment upon later.

On the rigid, lower plane, the temperature is fixed,

$$v_i = 0, \quad z = 0 \tag{2.8a}$$

$$\theta = 1, \quad z = 0. \tag{2.8b}$$

Finally, the Boussinesq equations for the bulk liquid have the form

$$P^{-1}\left(\frac{\partial v_i}{\partial t} + v_j v_{i,j}\right) = \sigma_{ij,j} + R^2 \theta k_i, \tag{2.9a}$$

$$\frac{\partial \theta}{\partial t} + v_i \theta_{,i} = \nabla^2 \theta, \tag{2.9b}$$

$$v_{i,i} = 0, \tag{2.9c}$$

where $k_i = (0, 1)_i$.

(*b*) *The basic state*

We can find a motionless basic state consistent with (2.2)–(2.9) of the form

$$\bar{\theta} = 1-z, \tag{2.10a}$$

$$\sigma_{ij} = G[(z-1)+\tfrac{1}{2}\delta(z-1)^2]\,\delta_{ij}, \tag{2.10b}$$

$$\bar{\eta} \equiv 0. \tag{2.10c}$$

This layer has a flat top, no motion, a linear temperature profile and isotropic, hydrostatic stress consistent with (2.7) and (2.8*b*).

(*c*) *The disturbance equations*

Disturbances of the basic state (2.10) satisfy the *nonlinear* system

$$P^{-1}\left(\frac{\partial v_i}{\partial t}+v_j v_{i,j}\right) = \sigma_{ij,j}+R^2\theta k_i, \tag{2.11a}$$

$$\frac{\partial\theta}{\partial t}+v_i\theta_{,i} = \nabla^2\theta+w, \tag{2.11b}$$

$$v_{i,i} = 0. \tag{2.11c}$$

The forms of σ, $\mathbf{n}$, $\mathbf{t}$, N, σ_{ij}, ϵ_{ij} and K are identical to those of (2.2), (2.3) and (2.5). The stress boundary dondition (2.4) then takes the form

$$\sigma_{ij}n_j = \frac{1}{C}\{K(\eta)\,[\sigma(\theta)+MC\eta]-G(\eta+\tfrac{1}{2}\delta\eta^2)\}\,n_i - M(\theta_{,k}-\eta_{,k})\,t_k t_i. \tag{2.11d}$$

The middle term in (2.11*d*) comes from the basic-state contribution to θ and, since η depends only on x and t, $\eta_{,i}t_i = \eta_x/N$. The kinematic condition remains unchanged in form:

$$\eta_t = Nv_i n_i \quad \text{on} \quad z = 1+\eta \tag{2.11e}$$

and the thermal condition on the free surface is

$$\theta_{,i}n_i = (1-N)/N \quad \text{on} \quad z = 1+\eta. \tag{2.11f}$$

Finally, we have the boundary conditions on the rigid plate:

$$v_i = \theta = 0 \quad \text{on} \quad z = 0. \tag{2.11g}$$

System (2.11) gives the basic equations that govern two-dimensional fluid motions driven jointly by buoyancy in the bulk and surface-tension gradients on the free surface. It is from these equations that statements of stability or instability may be made. While they form the focus of most of the remainder of the paper, we discuss generalizations to three dimensions, several bulk fields and more elaborate interfacial mechanics in § 7.

3. Energy stability theory

(*a*) *Preliminaries*

We consider system (2.11) which applies on the interval

$$0 \leqslant z \leqslant 1+\eta(x,t), \quad -\infty \leqslant x \leqslant \infty, \quad t \geqslant 0.$$

Since the range of x is infinite, we must consider that all disturbance quantities remain bounded for $|x| \to \infty$ so that the power integrals exist. We shall assume that v_i, θ, p and η are each X_0 periodic in x. [For disturbances that are Fourier-transformable or almost periodic, similar results apply; see Joseph (1976) for a discussion.] Apart from the X_0 periodicity and sufficient smoothness properties, v_i, θ, p and η are arbitrary; in particular, they have arbitrary amplitude.

The aim here is to obtain the power integrals from which we shall obtain a condition of certain stability. We shall find a stability limit R_E^2 such that, when $R < R_E$, the basic state is asymptotically stable in the mean. Here R_E depends on M, C and G. R_E emerges as the minimum eigenvalue of a set of nonlinear Euler–Lagrange equations obtained from the power integrals.

(*b*) *The power integrals*

We define the integral over the free surface of a quantity f as follows:

$$\int_{S(t)} f \equiv \int_0^{S_0(t)} f\,ds, \tag{3.1a}$$

where ds is an element of arc length along $S(t)$ and $S_0(t)$ is the arc length of the free surface in one period in x, i.e.

$$S_0(t) = \int_0^{X_0} N\,dx,$$

and N is given in (2.3*b*) as a functional of η. The equivalent of (3.1) over the corresponding fixed domain is given by

$$\int_X f \equiv \int_0^{X_0} f\,dx, \tag{3.1b}$$

where

$$S_0(t) = \int_X N. \tag{3.1c}$$

We thus have

$$\int_{S(t)} f = \int_X fN. \tag{3.2}$$

The 'volume' integral (in two dimensions) of a quantity f over one period in x is defined as follows:

$$\langle f \rangle \equiv \int_0^{X_0} \int_0^{1+\eta(x,t)} f(x,z,t)\,dz\,dx. \tag{3.3}$$

The first power integral, the mechanical-energy balance, is obtained by multiplying (2.12*a*) by v_i and integrating over the material volume defined in (3.3) and shown in figure 2, using the divergence theorem and the continuity equation (2.11*c*). The result involves the term J,

$$J \equiv \langle v_i \sigma_{ij,j} \rangle.$$

It may be shown that

$$J = \int_{S(t)} v_i \sigma_{ij} n_j - \tfrac{1}{2} \langle \epsilon_{ij} \epsilon_{ij} \rangle,$$

where we have used the definition (2.5*b*), the no-slip condition on the solid, and the X_0 periodicity of the disturbances. The interfacial stress condition (2.11*d*) allows us to write

$$J = \int_{S(t)} \left[\frac{1}{C} [K(\eta)\,\sigma(\theta) - G(\eta + \tfrac{1}{2}\delta\eta^2) + MCK(\eta)\,\eta]\, v_i n_i - M v_j t_j t_i (\theta_{,i} - \eta_{,i}) \right] - \tfrac{1}{2} \langle \epsilon_{ij} \epsilon_{ij} \rangle. \tag{3.4a}$$

We can thus write down the mechanical-energy balance:

$$\tfrac{1}{2}P^{-1}\frac{d}{dt}\langle v_i v_i\rangle = R^2\langle w\theta\rangle + J. \tag{3.4b}$$

This generalizes the equation given by Davis (1969) to arbitrary surface displacements and is equivalent to that obtained by Dussan V. (1975) when specialized to constant surface tension, $M = 0$.

We can obtain the second power integral, the 'entropy balance', by multiplying (2.11*b*) by θ and integrating. If the boundary conditions on the solid and the X_0 periodicity are used, we obtain the equation

$$\frac{1}{2}\frac{d}{dt}\langle\theta^2\rangle = -\langle\theta_{,i}\theta_{,i}\rangle + \langle w\theta\rangle + \int_{S(t)}\theta(1-N)/N. \tag{3.5}$$

The power integrals (3.4) and (3.5) are tightly coupled to the surface displacement η, both through the appearance of η in the definition of the volume integral and through the explicit appearance of η in the boundary term in J. This coupling suggests that the kinematic condition (2.11*e*) be treated as a *surface evolution equation* for η and that a *surface power integral* be defined. Several choices for weight function can be made. The following seems particularly convenient: multiply (2.11*e*) by $(\eta + \frac{1}{2}\delta\eta^2)/N$ and integrate over $S(t)$. We find that

$$\frac{1}{2}\frac{d}{dt}\int_{S(t)}(\eta^2 + \tfrac{1}{3}\delta\eta^3)\,N^{-1} = \int_{S(t)}(\eta + \tfrac{1}{2}\delta\eta^2)\,v_i n_i. \tag{3.6}$$

This choice is suggested by the appearance of the right-hand side of (3.6) in J of (3.4) so we can use this form to *eliminate the explicit dependence on* G in the energy theory to come. In other words, we wish to form a generalized energy functional E as the sum

$$\langle\tfrac{1}{2}P^{-1}v_i v_i\rangle + \lambda_1\langle\tfrac{1}{2}\theta^2\rangle + \lambda_2\int_{S(t)}\tfrac{1}{2}(\eta^2 + \tfrac{1}{3}\delta\eta^3)/N.$$

This linear combination must be positive definite. When $|\eta| < 3/\delta$, as it must be under the Boussinesq approximation, then the surface integral above is positive. Hence, E is positive if $\lambda_1, \lambda_2 > 0$. λ_1 and λ_2 are called linking parameters (Joseph 1965) and should be chosen optimally to give the largest parameter region of certain stability.

In accordance with the discussion above we choose $\lambda_2 = G/C$. The parameter G is thus suppressed explicitly but still has an implicit effect through the form of the resulting equations. This is important since we know from linear theory (Smith 1966; Zeren & Reynolds 1972) that, for small to moderate capillary numbers C, the Bond number term damps the long-wave modes described by Scriven & Sternling (1964) and the interfacial modes of Gummerman & Homsy (1974). Thus, for $G = 0$ the layer is always unstable to long waves according to linear theory. The results that we shall report in § 4 are in fact for modes of moderate length and so are not expected to be severely affected by such gravity-wave effects. In any case, the choice of the linking parameter $\lambda_2 = G/C$ is one of convenience and so presumably it is not an optimal choice. (One could, of course, elect to make λ_2 general.) If, indeed, this choice is non-optimal, then the stability limits so obtained may be more conservative than need be.

We thus define† the positive-definite 'generalized energy' functional E,

$$E = \tfrac{1}{2}\langle P^{-1}v_i v_i + \lambda^2 R^2\theta^2\rangle + (G/C)\int_{S(t)} (\eta^2 + \tfrac{1}{2}\delta\eta^3)\, N^{-1}, \tag{3.7a}$$

where $\lambda > 0$ is the remaining linking parameter which we can choose to give the largest region of stability available in the theory. We can then define the additional functions that appear in the 'generalized energy' balance as follows:

$$\mathscr{P} = 2\langle w\phi\rangle \tag{3.7b}$$

$$\mathscr{D} = \tfrac{1}{2}\langle \epsilon_{ij}\epsilon_{ij}\rangle + \langle \phi_{,i}\phi_{,i}\rangle \tag{3.7c}$$

$$\mathscr{S} = \int_{S(t)} \left\{\frac{1}{C} K(\sigma + MC\eta)\, v_i n_i - M v_j t_j (\phi_{,i}/R\lambda - \eta_{,i})\, t_i + R\lambda\phi(1-N)\, N^{-1}\right\}, \tag{3.7d}$$

where $\phi \equiv R\lambda\theta$. The resulting *energy-evolution equation* is thus

$$\frac{dE}{dt} = -\mathscr{D} + \mathscr{R}\mathscr{P} + \mathscr{S}, \tag{3.8}$$

where

$$\mathscr{R} \equiv \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) R. \tag{3.9}$$

It is from this energy-evolution equation that statements of stability may be made.

(c) *The energy theory*

We follow the approach of Davis & von Kerczek (1973) and seek values $R_E(M, C)$ such that the condition $R < R_E$ guarantees asymptotic stability in the mean of the basic state. To this end, divide the generalized energy equation (3.8) by E and consider the maximum‡ of this ratio as follows:

$$\frac{1}{E}\frac{dE}{dt} = (\mathscr{R}\mathscr{P} - \mathscr{D} + \mathscr{S})/E \leqslant \nu(\mathscr{R}) \quad \text{for all } t, \tag{3.10a}$$

where

$$\nu(\mathscr{R}) = \max_{0\leqslant t\leqslant t_0} \max_{\mathscr{H}} \{(\mathscr{R}\mathscr{P} - \mathscr{D} + \mathscr{S})/E\} \tag{3.10b}$$

and

$$\mathscr{H} = \left\{ \begin{array}{l} v_i, \phi, \eta \,|\, v_{i,i} = 0; \quad \displaystyle\int_X \eta = 0; \quad v_i = \phi = 0, \quad z = 0; \\ v_i, \phi, \eta \text{ are } X_0 \text{ periodic in } x \text{ and sufficiently smooth} \end{array} \right\} \tag{3.10c}$$

Since it follows from (3.10) that

$$E(t) \leqslant E(0)\, e^{\nu(\mathscr{R})\, t},$$

a condition for stability is that $\mathscr{R} < \mathscr{R}_E$, where $\mathscr{R}_E$ is the smallest value of $\mathscr{R}$ such that $\nu(\mathscr{R}_E) = 0$. This translates into finding the Euler–Lagrange equations for

$$\delta(\mathscr{R}\mathscr{P} - \mathscr{D} + \mathscr{S}) = 0 \tag{3.11}$$

against admissible functions defined by $\mathscr{H}$. In this class, conservation of mass implies that volume is conserved so that the condition $\int_X \eta = 0$ is required. If we call β and

† The ability to explicitly eliminate G for the right-hand side of (3.8) is directly related to the choice of surface norm, equations (3.6)–(3.7); this furnishes one justification for its use.

‡ We *assume* that $\nu(\mathscr{R}) \not\equiv 0$. If $\nu(\mathscr{R})$ were identically zero, then the condition $R < R_E$ would still be the stability condition for $dE/dt < 0$ but this would not guarantee that $E \to 0$ as $t \to \infty$.

$2p(x,z,t)$ the Lagrange multipliers associated with this condition and $v_{i,i}=0$, respectively, the unconstrained variational principle associated with form (3.11) is given as follows:

$$\delta\left(\mathscr{R}\mathscr{P}-\mathscr{D}+\mathscr{S}+\langle 2pv_{i,i}\rangle+\beta\int_X \eta\right)=0. \tag{3.12}$$

In taking variations, we shall require the following formulae:

$$\delta\mathscr{P}=2\langle w\delta\phi+\phi\delta w\rangle+2\int_X w\phi\delta\eta; \tag{3.13a}$$

$$\delta\mathscr{D}=-2\langle\epsilon_{ij,j}\delta v_i+\nabla^2\phi\delta\phi\rangle+\int_X\{2N\epsilon_{ij}n_j\delta v_i+2N\phi_{,i}n_i\delta\phi+(\tfrac{1}{2}\epsilon_{ij}\epsilon_{ij}+\phi_{,i}\phi_{,i})\,\delta\eta\}; \tag{3.13b}$$

$$\delta\langle 2pv_{i,i}\rangle=2\langle p_{,i}\delta v_i+v_{i,i}\delta p\rangle+2\int_X(pNn_i\delta v_i+pv_{i,i}\delta\eta); \tag{3.13c}$$

$$\delta\left(\beta\int_X\eta\right)=\beta\int_X\delta\eta+\delta\beta\int_X\eta. \tag{3.13d}$$

It is convenient in calculating $\delta\mathscr{S}$ to define the directional derivative of a surface quantity q as follows:

$$q'\equiv\partial q/\partial x+\eta_x\,\partial q/\partial z.$$

This allows us to perform integration by parts along the surface $S(t)$ since q' is proportional to the tangential derivative: $q'=N\,\partial q/\partial s$.

Care must be taken in order to include the variations $\delta\eta$. For example, for a functional F defined on the surface, we have

$$\begin{aligned}\delta\int_{S(t)}F[\eta',\theta(x,1+\eta,t)]\,ds&=\delta\int_X NF[\eta',\theta(x,1+\eta,t)]\,dx\\&=\int_X\{NF_\theta\delta\theta+[NF_\theta\theta_z-(NF_{\eta'})']\,\delta\eta\}\,dx.\end{aligned}$$

In computing $\delta\mathscr{S}$, we have need for the following formulae:

$$\delta K=[N^{-3}(\delta\eta)']'; \tag{3.14a}$$

$$\delta(Nt_j)=(0,1)_j\,(\delta\eta)'; \tag{3.14b}$$

$$\delta(Nn_j)=-(1,0)_j\,(\delta\eta)'; \tag{3.14c}$$

$$\delta t_j=n_j(\delta\eta)'/N^2; \tag{3.14d}$$

$$\delta\eta_j=-t_j(\delta\eta)'/N^2. \tag{3.14e}$$

Hence,

$$\int_X F\delta K=\int_X(F'N^{-3})'\,\delta\eta, \tag{3.15a}$$

$$\int_X Fv_j\delta(Nn_j)=\int_X(Fu)'\,\delta\eta \tag{3.15b}$$

and

$$\int_X Fv_j\delta(Nt_j)=-\int_X(Fw)'\,\delta n. \tag{3.15c}$$

After much manipulation, we find that

$$\delta\mathscr{S}=\int_X(V_i\delta v_i+\Phi\delta\phi+H\delta\eta), \tag{3.16a}$$

where

$$V_i = \frac{1}{C}K(\sigma + MC\eta)\,Nn_i - M(\phi'/R\lambda - \eta_x)\,t_i; \tag{3.16b}$$

$$\Phi = Mv_j' t_j/R\lambda + \lambda R(1-N); \tag{3.16c}$$

$$\begin{aligned} H = \frac{1}{C}\{&[[(\sigma + MC\eta)\,v_i n_i N]'\,N^{-3}]' + [K(\sigma + MC\eta)\,u]' + [K(\sigma + MC\eta)\,v_i n_i N]_z\} \\ &+ R\lambda[(\phi\eta_x N^{-1})' + \phi_z(1-N)] - M[((\phi'/R\lambda - \eta_x)\,v_i t_i)_z - (v_i n_i N^{-2}(\phi'/R\lambda - \eta_x))' \\ &+ (v_i t_i)' - (\phi_z v_j t_j)'/R\lambda - K n_i v_i N] \end{aligned} \tag{3.16d}$$

and

$$\sigma = 1 - \frac{MC}{R\lambda}\phi. \tag{3.16e}$$

The Euler–Lagrange equations and natural boundary conditions may now be written down by combining (3.12), (3.13) and (3.16). The result is as follows:

$$\sigma_{ij,j} + \mathscr{R}\phi k_i = 0, \tag{3.17a}$$

$$\nabla^2\phi + \mathscr{R}w = 0, \tag{3.17b}$$

$$v_{i,i} = 0, \tag{3.17c}$$

$$v_i = \phi = 0, \quad z = 0; \tag{3.17d}$$

$$C\sigma_{ij}n_j = \tfrac{1}{2}K(\sigma + MC\eta)\,n_i - \tfrac{1}{2}MC(\phi'/R\lambda - \eta_x)\,t_i/N, \quad z = 1+\eta, \tag{3.17e}$$

$$\phi_{,i}n_i = \frac{1}{2}\frac{M}{R\lambda}v_i' t_i/N + R\lambda(1-N), \quad z = 1+\eta. \tag{3.17f}$$

$$\begin{aligned} C[2\mathscr{R}w\phi - \tfrac{1}{2}\epsilon_{ij}\epsilon_{ij} - \phi_{,i}\phi_{,i} + \beta] + &[[(\sigma + MC\eta)\,v_i n_i N]'\,N^{-3}]' + [K(\sigma + MC\eta)\,u]' \\ &+ [K(\sigma + MC\eta)\,v_i n_i N]_z + R\lambda C[(\phi\eta_x N^{-1})' + \phi_z(1-N)] - MC[((\phi'/R\lambda - \eta_x)\,v_i t_i)_z \\ &- (v_i n_i N^{-2}(\phi'/R\lambda - \eta_x))' + (v_i t_i)' - (\phi_z v_j t_j)'/R\lambda - K v_i n_i N], \end{aligned} \tag{3.17g}$$

with

$$\int_X \eta = 0. \tag{3.17h}$$

Here (3.17*a, b, c*) respectively result from the bulk variations δv_i, $\delta\phi$ and δp respectively. Equations (3.17*e, f, g, h*) respectively result from surface variations δv_i, $\delta\phi$, $\delta\eta$ and $\delta\beta$.

The system (3.17) constitutes a free-boundary problem. However, in contrast to all previous energy analysis in hydrodynamics (except Joseph & Hung 1971) for which the Euler–Lagrange systems are linear, system (3.17) is nonlinear. (The bulk equations (3.17*a, b, c*) are linear but the (natural) boundary conditions are nonlinear.†) This nonlinearity indicates that energy theory of free-surface problems should lead to *conditional stability* results, i.e. the stability limits should depend not only upon the parameters of the problem (here M, C, G) but upon the disturbance amplitude. When axisymmetric Taylor vortices are examined by energy theory (Joseph & Hung 1971), a conditional result emerges. The only interfacial example in the literature is in Rayleigh–Taylor instability where Dussan V. (1975) obtained threshold amplitudes for both stability and instability. These issues will be discussed more fully in § 7.

† This is a structure reminiscent of inviscid water waves.

4. Energy theory for small capillary number

(a) The perturbation expansion

The Euler–Lagrange system (3.17) is strongly nonlinear. Hence, apart from a numerical study, limiting cases must be attacked in order to gain insight into the meaning of the energy limit.

It has been observed by Scriven & Sternling (1964) and Smith (1966) that for most fluids under most conditions the capillary number C is quite small, commonly ranging from 10^{-6} to 10^{-2}. For example, a layer of pure water 1 cm deep at 20 °C has $C = 2 \times 10^{-7}$. It is only under special circumstances, such as for thin layers with low interfacial tensions, that C approaches unity. Hence, we shall examine system (3.17) for $C \to 0$. In so doing we shall recover at first order the previous energy results of an analysis (Davis 1969) that presumed a flat interface. We shall further obtain the first non-zero correction of the energy limit R_E for small C. In so doing we obtain a connexion between the conditional limits and the physical variable C. In order to make a useful comparison, analogous linear theory results must be obtained; these are given in § 5.

We fix M and G and seek a regular perturbation solution of system (3.17) in powers of C. We write

$$(v_i, \phi, p, \eta) \sim (v_i^{(0)}, \phi^{(0)}, p^{(0)}, \eta^{(0)}) + C(v_i^{(1)}, \phi^{(1)}, p^{(1)}, \eta^{(1)}) + \ldots \tag{4.1a}$$

and seek the energy limit R_E^2 of the Rayleigh number R^2. As such, we must likewise write

$$R = R^{(0)} + CR^{(1)} + O(C^2) \tag{4.1b}$$

and

$$\lambda = \lambda^{(0)} + C\lambda^{(1)} + O(C^2), \tag{4.1c}$$

$$\beta = \beta^{(0)} + C\beta^{(1)} + O(C^2), \tag{4.1d}$$

where λ is the linking parameter and β is the Lagrange multiplier introduced to constrain η to have a zero mean. In terms of (4.1 c, d) and (3.9) we have

$$\mathscr{R} = \mathscr{R}^{(0)} + C\mathscr{R}^{(1)} + O(C^2). \tag{4.1e}$$

$\mathscr{R}$ is retained in the expansion, since it appears naturally below as the eigenvalue of the first-order problem.

If we substitute forms (4.1) into system (3.17) and equate to zero coefficients of like powers of C, we obtain a set of problems to be solved sequentially. This set is defined on the domain $[0, X_0] \times [0, 1+\eta]$. It is convenient to solve them on the fixed domain $[0, X_0] \times [0, 1+\eta^{(0)}]$. The interfacial conditions must then each be referred to the unperturbed position $z = 1 + \eta^{(0)}$ in the usual way.

(b) The first-order problem

At leading order, we have

$$\sigma_{ij,j}^{(0)} + \mathscr{R}^{(0)} \phi^{(0)} k_i = 0, \tag{4.2a}$$

$$\nabla^2 \phi^{(0)} + \mathscr{R}^{(0)} w^{(0)} = 0, \tag{4.2b}$$

$$v_{i,i}^{(0)} = 0, \tag{4.2c}$$

$$v_i^{(0)} = \phi^{(0)} = 0 \quad \text{on} \quad z = 0; \tag{4.2d}$$

$$K^{(0)} = 0 \quad \text{on} \quad z = 1+\eta^{(0)}, \tag{4.3a}$$

$$\sigma_{ij}^{(0)} n_j^{(0)} t_i^{(0)} = -\tfrac{1}{2} M(\phi^{(0)\prime}/R^{(0)}\lambda^{(0)} - \eta_x^{(0)})/N^{(0)} \quad \text{on} \quad z = 1+\eta^{(0)}, \tag{4.3b}$$

$$\phi_{,i}^{(0)} n_i^{(0)} = \tfrac{1}{2} M v_i^{(0)\prime} t_i^{(0)}/R^{(0)}\lambda^{(0)} N^{(0)} + \lambda^{(0)} R^{(0)}(1-N^{(0)})/N^{(0)} \quad \text{on} \quad z = 1+\eta^{(0)}, \tag{4.3c}$$

$$[(1+\eta^{(0)})(v_i^{(0)} n_i^{(0)} N^{(0)})' N^{(0)-3}]' + (1+\eta^{(0)}) K^{(0)} v_{i,z}^{(0)} n_i^{(0)} N^{(0)} + [(1+\eta^{(0)}) K^{(0)} u^{(0)}]' + K^{(0)} n_i^{(0)} v_i^{(0)} N^{(0)} = 0 \quad \text{on} \quad z = 1+\eta^{(0)}, \tag{4.3d}$$

$$\int_X \eta^{(0)} = 0; \tag{4.3e}$$

$$\mathscr{R}^{(0)} = \tfrac{1}{2} R^{(0)} \left(\lambda^{(0)} + \frac{1}{\lambda^{(0)}}\right), \tag{4.3f}$$

and now a prime denotes $\partial/\partial x + \eta_x^{(0)}\,\partial/\partial z$. Equations (4.3*a*) and (4.3*b*) are obtained from condition (3.17*e*) by resolving into components normal and tangential to $S(t)$, respectively.

Systems (4.2) and (4.3) are inherently *nonlinear* since all the interfacial conditions are applied on $z = 1+\eta^{(0)}$ where $\eta^{(0)}$ is unknown. However, in our problem there is a decoupling that makes the solution elementary. To see this, we write condition (4.3*a*) in the form

$$\eta_{xx}^{(0)}/N^{(0)3} = 0.$$

Since $\eta^{(0)}$ must be X_0 periodic and by equation (4.3*e*) of zero mean, it follows that

$$\eta^{(0)} \equiv 0. \tag{4.4}$$

Hence, the limit $C \to 0$ is equivalent to large mean surface tension and so at leading order there is a zero surface displacement. Scriven & Sternling (1964) have made the same observation for the *linear stability* theory. The same result is seen to apply in energy theory, a feature only *assumed* by Davis (1969). The present work justifies this assumption by showing that displacements $\eta^{(0)}$ that are *only kinematically constrained* (i.e. belong to class $\mathscr{H}$) likewise satisfy the dynamic condition (4.4).

As a result of the expansions (4.1) and result (4.4), we have the

$$\mathbf{n} = \mathbf{n}^{(0)} + C\mathbf{n}^{(1)} + O(C^2), \tag{4.5a}$$

$$\mathbf{t} = \mathbf{t}^{(0)} + C\mathbf{t}^{(1)} + O(C^2), \tag{4.5b}$$

$$N = N^{(0)} + O(C^2), \tag{4.5c}$$

where

$$\mathbf{n}^{(0)} = (0,1), \quad \mathbf{t}^{(0)} = (1,0), \quad N^{(0)} = 1, \tag{4.5d}$$

$$\mathbf{n}^{(1)} = -\eta_x^{(1)}\mathbf{t}^{(0)}, \quad \mathbf{t}^{(1)} = \eta_x^{(1)}\mathbf{n}^{(0)}. \tag{4.5e}$$

If we use results (4.4) and (4.5*d*), then condition (4.3*d*) recovers the kinematic condition for flat interfaces,

$$w^{(0)} = 0, \quad z = 1. \tag{4.6}$$

If we use results (4.4) and (4.6), then conditions (4.3*b*, *c*) become

$$\sigma_{ij}^{(0)} n_j^{(0)} t_i^{(0)} + \frac{1}{2}\frac{M}{\lambda^{(0)} R^{(0)}}\phi_x^{(0)} = 0, \quad z = 1, \tag{4.7a}$$

$$\phi_{,i}^{(0)} n_i - \frac{1}{2}\frac{M}{\lambda^{(0)} R^{(0)}} u_x^{(0)} = 0, \quad z = 1 \tag{4.7b}$$

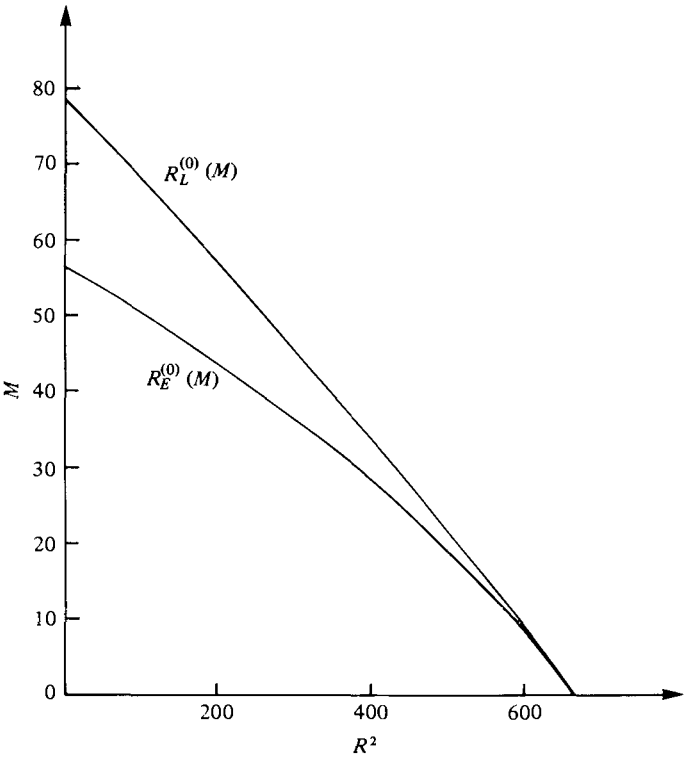

FIGURE 3. The stability boundaries for buoyancy–capillary instability of a layer having a non-deformable ($C = 0$) planar interface. When $R < R_L^{(0)}(M)$, infinitesimal non-oscillatory disturbances decay according to the linear theory of Nield (1964). When $R < R_E^{(0)}(M)$ disturbances of arbitrary amplitude decay according to the energy theory of Davis (1969).

The system (4.2), (4.6) and (4.7) is equivalent to the two-dimensional version of that of Davis (1969), derived from the restricted variational principle with $\eta \equiv 0$. (It is important to note that the first-order problem is nonlinear because of (4.3a). However, its particularly simple solution leads to a system which *is* linear.) As such, the system must be self-adjoint; the appropriate scalar product $(\psi^{(1)}, \psi^{(2)})$ of $\psi^{(1)}$ and $\psi^{(2)}$ is given by

$$(\psi^{(1)}, \psi^{(2)}) \equiv \langle v_i^{(1)} v_i^{(2)} + \phi^{(1)}\phi^{(2)} \rangle, \tag{4.8}$$

where

$$\psi^{(1)} = \begin{Bmatrix} v_i^{(1)} \\ \phi^{(1)} \end{Bmatrix}, \quad \psi^{(2)} = \begin{Bmatrix} v_i^{(2)} \\ \phi^{(2)} \end{Bmatrix}.$$

Furthermore, it will be useful in what follows to note that (4.2), (4.6) and (4.7) have separable solutions whose x variation is proportional to $\exp(i\alpha x)$.

The energy limit $R_E^{(0)}$ is given by the above system (Davis 1969, figure 1) as

$$R_E^{(0)}(M) = \min_{\alpha} \max_{\lambda} R^{(0)}(\lambda, \alpha, M), \tag{4.9}$$

where α is the wavenumber in the x direction. It is seen (Joseph 1965; Davis 1969) that $\lambda^{(0)} = 1$ for $M = 0$ and increases monotonically with M. We show these results together with those of linear theory (Nield 1964) in figure 3. We shall refer to them in § 6 below.

(c) *The second-order problem*

Consider the problem for the variables at $O(C)$. These are

$$\sigma^{(1)}_{ij,j} + \mathscr{R}^{(0)}\phi^{(1)}k_i = -\mathscr{R}^{(1)}\phi^{(0)}k_i, \tag{4.10a}$$

$$\nabla^2\phi^{(1)} + \mathscr{R}^{(0)}w^{(1)} = -\mathscr{R}^{(1)}w^{(0)}, \tag{4.10b}$$

$$v^{(1)}_{i,i} = 0, \tag{4.10c}$$

$$v^{(1)}_i = \phi^{(1)} = 0 \quad \text{on} \quad z = 0, \tag{4.10d}$$

where

$$\mathscr{R}^{(1)} = \frac{1}{2}\left\{\left(\lambda^{(0)} + \frac{1}{\lambda^{(0)}}\right)R^{(1)} + R^{(0)}\left(1 - \frac{1}{\lambda^{(0)2}}\right)\lambda^{(1)}\right\}. \tag{4.10e}$$

The results of the normal component of (3.17e) is that

$$\eta^{(1)}_{xx} = 2[\sigma^{(0)}_{ij}n^{(0)}_j u^{(0)}_i]_{z=1},$$

so that in terms of normal modes we have

$$\eta^{(1)} = -\frac{2}{\alpha^2}[\sigma^{(0)}_{ij}n^{(0)}_j n^{(0)}_i]_{z=1}. \tag{4.11}$$

The displacement $\eta^{(1)}$ may be calculated from the solutions of the $O(1)$ problem. It is found that $\eta^{(1)} \not\equiv 0$, and in particular $\eta^{(1)} = \eta_1 \exp(i\alpha x)$. If we use (4.11), then the tangential component of (3.17e) and (3.17f) becomes at $O(C)$:

$$\sigma^{(1)}_{ij}n^{(0)}_j t^{(0)}_i + \frac{1}{2}\frac{M}{\lambda^{(0)}R^{(0)}}\phi^{(1)}_x = A_1 + R^{(1)}B_1, \quad z = 1, \tag{4.12a}$$

$$\phi^{(1)}_{,i}n^{(0)}_i - \frac{1}{2}\frac{M}{\lambda^{(0)}R^{(0)}}u^{(1)}_x = C_1 + R^{(1)}D_1, \quad z = 1, \tag{4.12b}$$

$$w^{(1)}_{xx} = \hat{E}_1, \quad z = 1$$

or

$$w^{(1)} = E_1, \quad z = 1, \tag{4.12c}$$

where A_1, B_1, C_1, D_1, E_1 and $\hat{E}_1$ are given in the appendix and depend on $O(1)$ solutions, $R^{(0)}$, $\lambda^{(0)}$ and $\lambda^{(1)}$. Equations (4.5e) have been used. The constant $\beta^{(0)}$ in $\hat{E}_1$ is chosen so that E_1 is X_0 periodic and has a zero mean.

The inhomogeneous system (4.10) and (4.12) only has solutions if a solvability condition, using the scalar product (4.8), holds. We obtain this condition by multiplying (4.12a) by $v^{(0)}_i$, (4.12b) by $\phi^{(0)}$, adding, integrating over one cell, and using the divergence theorem and the boundary conditions. The result is

$$\int_X \sigma^{(1)}_{ij}n^{(0)}_j v^{(0)}_i - \int_X \sigma^{(0)}_{ij}n^{(0)}_j v^{(1)}_i + \int_X \phi^{(1)}_{,i}n^{(0)}_i\phi^{(0)} - \int_X \phi^{(0)}_{,i}n^{(0)}_i\phi^{(1)} = -2\mathscr{R}^{(0)}\langle\phi^{(0)}w^{(0)}\rangle. \tag{4.13}$$

Here $\int_X$ denotes the integral over x, $0 \leqslant x \leqslant X_0$, along the undisturbed surface $z = 1$. We now use the conditions (4.5), (4.7) and (4.12a, b, c) and relations (4.5e) after observing that the form, say, $\sigma^{(1)}_{ij}n^{(0)}_j v^{(0)}_i = (\sigma^{(1)}_{ij}n^{(0)}_j n^{(0)}_i)\,w^{(0)} + (\sigma^{(1)}_{ij}n^{(0)}_j t^{(0)}_i)\,u^{(0)}$. Condition (4.13) then becomes

$$R^{(1)}\left\{\left(\lambda^{(0)} + \frac{1}{\lambda^{(0)}}\right)\langle w^{(0)}\phi^{(0)}\rangle + \int_X u^{(0)}B_1 + \int_X \phi^{(0)}D_1\right\}$$

$$= -\left\{\lambda^{(1)}\left(1 - \frac{1}{\lambda^{(0)2}}\right)R^{(0)}\langle w^{(0)}\phi^{(0)}\rangle + \int_X u^{(0)}A_1 - \int_X (\sigma^{(0)}_{ij}n^{(0)}_j n^{(0)}_i)\,E_1 + \int_X \phi^{(0)}C_1\right\}. \tag{4.14}$$

Since the first-order fields and A_1, B_1, etc., are known, this relation determines $R^{(1)}$.

M	$\alpha^{(0)}$	$\lambda^{(0)}$	$R_E^{(0)2}$	$R_E^{(0)2}$	$R_E^{(1)} \times 10^{-3}$
0	2·10	1·00	25·9	669	0·000
10	2·08	1·05	24·2	586	0·205
20	2·08	1·15	22·1	490	0·454
25	2·08	1·30	20·9	437	0·582
30	2·10	1·40	19·5	380	0·732
35	2·12	1·55	17·9	319	0·903
40	2·14	1·75	15·9	254	1·14
45	2·16	2·15	13·6	184	1·32
50	2·18	2·85	10·4	109	1·65

TABLE 1. Results of numerical calculations for the energy theory of buoyancy–thermocapillary layers with $C \to 0$. $\alpha^{(0)}$ and $\lambda^{(0)}$ are the optimal wavenumber and linking parameter, respectively. The critical Rayleigh number of energy theory is given by $R_E^{(0)2} + 2R_E^{(0)}R_E^{(1)}C + O(C^2)$.

Examination of (4.14) shows that, in order for $R^{(1)}$ to be non-zero, either $\lambda^{(0)} \neq 1$ or the surface integrals over x, arising from the boundary conditions, must not vanish. The formulae for A_1, E_1 and C_1 given in the appendix show that these expressions contain terms linear in the first-order fields or $\eta^{(1)}$ and proportional to M, or terms quadratic in the first-order fields and $\eta^{(1)}$. These quadratic terms give no contribution to $R^{(1)}$, since they are proportional to $e^{2i\alpha x}$ (with zero mean), and hence are orthogonal to the first-order fields, cf. (4.14). The only terms which contribute to $R^{(1)}$ are those proportional to M. With the aid of the expressions for A_1, B_1, etc., in the appendix, we find that (4.14) reduces to

$$R^{(1)} = -\left\{\lambda^{(1)}\left[R^{(0)}\left(1-\frac{1}{\lambda^{(0)2}}\right)\langle w^{(0)}\phi^{(0)}\rangle + \frac{M}{R^{(0)}\lambda^{(0)2}}\int_X w_z^{(0)}\phi^{(0)}\right]\right.$$
$$\left. - \frac{2M}{\alpha^2}\int_X w_z^{(0)}\sigma_{ij}^{(0)}n_j^{(0)}n_i^{(0)}\right\}\left\{\left(\lambda^{(0)-1}+\lambda^{(0)}\right)\langle w^{(0)}\phi^{(0)}\rangle + \frac{M}{R^{(0)2}\lambda^{(0)}}\int_X w_z^{(0)}\phi^{(0)}\right\}^{-1}. \quad (4.15)$$

It is possible to show that the bracketed term multiplying $\lambda^{(1)}$ is $[\partial R^{(0)}/\partial\lambda]_{\lambda=\lambda^{(0)}}$ and is therefore zero since $\lambda^{(0)}$ is chosen to maximize $R^{(0)}$. In this case $\lambda^{(1)}$ is undetermined and $R^{(1)}$ is given by

$$R^{(1)} = \frac{2M}{\alpha^2}\left\{\int_X w_z^{(0)}\sigma_{ij}^{(0)}n_j^{(0)}n_i^{(0)}\right\}\left\{\left(\lambda^{(0)}+\frac{1}{\lambda^{(0)}}\right)\langle w^{(0)}\phi^{(0)}\rangle + \frac{M}{R^{(0)2}\lambda^{(0)}}\int_X \phi^{(0)}w_z^{(0)}\right\}^{-1}. \quad (4.16)$$

The denominator in (4.16) is positive since it is a normalization constant of system (4.2), (4.6) and (4.7). Hence the sign of $R^{(1)}$, given by (4.16), is determined by the product on the surface of $w_z^{(0)}$ and the normal stress $\sigma_j^{(0)}n_j^{(0)}n_i^{(0)}$. In the case that above a rising current (i.e. $w_z^{(0)} < 0$) the normal stress is positive (negative), $R^{(1)}$ is negative (positive).

It is easy to show that the most dangerous wavenumber is $\alpha^{(0)} + O(C)$, where $\alpha^{(0)}$ is the critical wavenumber of the $O(1)$ problem. Hence, $R_E^{(1)}$ may be evaluated using $\alpha = \alpha^{(0)}$. We have computed $R_E^{(1)}$ using (4.16); the details of the numerical method are given in the appendix. The results are given in table 1 and figure 4. We note that for $0 \leqslant M \leqslant M_E \approx 56$, $R^{(1)} > 0$ indicating that the energy limit is *raised* by allowing weak surface deflexions. We discuss this in more detail in § 6 where we compare with the analogous results for the linear stability problem.

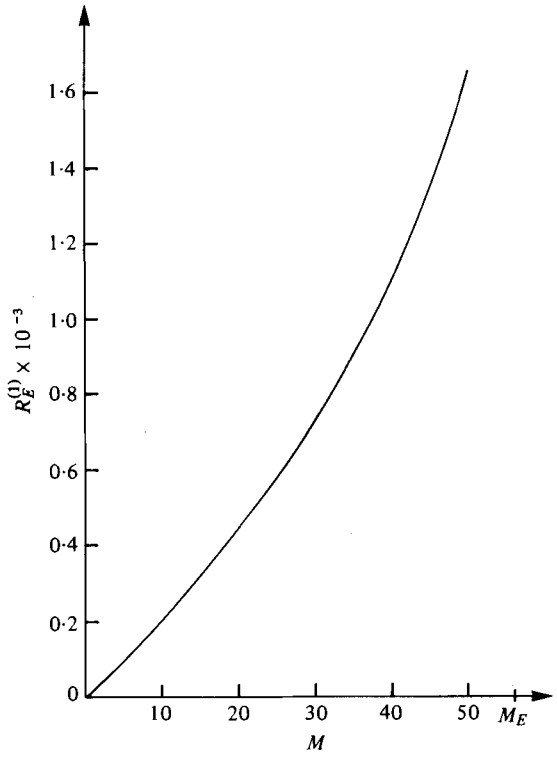

FIGURE 4. The function $R_E^{(1)}(M)$ for the present two-dimensional energy theory of buoyancy–thermocapillary convection in layers. $R_E^{(1)}$ is the $O(C)$ correction to the $R_E^{(0)}$.

We conclude from (4.14) that without thermocapillarity, $M = 0$ (in which case $\lambda^{(0)} = 1$), we have $R^{(1)} = 0$. This result depends on our choice of the thermal condition on the free surface and may not hold for more general cooling laws.

5. Linear stability theory for small C

In order to assess the global stability of buoyancy–thermocapillary layers we must have linear theory† results for the same parameter ranges as we do for energy theory. Although Smith (1966) and implicitly Zeren & Reynolds (1972) and Nield (1977) give small C results, it is useful to develop explicit perturbation results for direct comparison with those of § 4. We discover the interesting result that surface deflexions in thermocapillary convection may either be stabilizing or destabilizing compared to the case of no deflexion.

The linear stability problem is governed by the linearized version of system (2.12) as follows:

$$P^{-1}v_{i,t} = \sigma_{ij,j} + R^2\theta k_i, \tag{5.1a}$$

$$\theta_t = \nabla^2\theta + w, \tag{5.1b}$$

$$v_{i,i} = 0, \tag{5.1c}$$

† This linear theory, though given for convenience below in terms of two-dimensional disturbances, is valid for three-dimensional disturbances as well.

with
$$v_i = \theta = 0, \quad z = 1, \tag{5.1d}$$
$$C\sigma_{ij} n_j^{(0)} n_i^{(0)} = \eta_{xx} - G\eta, \quad z = 1, \tag{5.1e}$$
$$\sigma_{ij} n_j^{(0)} t_i^{(0)} = -M(\theta_{,i} - \eta_{,i}) t_i^{(0)}, \quad z = 1, \tag{5.1f}$$
$$\eta_t = v_i n_i^{(0)}, \quad z = 1, \tag{5.1g}$$
$$\theta_{,i} n_i^{(0)} = 0, \quad z = 1. \tag{5.1h}$$

We set $R\theta = \phi$, and expand the field variables (v_i, ϕ, η) and the eigenvalue R in powers of C.

(a) *The first-order problem*

The leading term of (5.1*e*) is
$$\eta_{xx}^{(0)} - G\eta^{(0)} = 0. \tag{5.2}$$
It is easy to show that the only X_0-periodic solution of (5.2) having a zero mean is
$$\eta^{(0)} \equiv 0, \tag{5.3}$$
a result due to Scriven & Sternling (1964). Using this result, the leading-order governing system, *assuming stationary onset*, is as follows:
$$\sigma_{ij,j}^{(0)} + R^{(0)} \phi^{(0)} k_i = 0, \tag{5.4a}$$
$$\nabla^2 \phi^{(0)} + R^{(0)} w^{(0)} = 0, \tag{5.4b}$$
$$v_{i,i}^{(0)} = 0, \tag{5.4c}$$
$$v_i^{(0)} = \phi^{(0)} = 0, \quad z = 0, \tag{5.4d}$$
$$\sigma_{ij}^{(0)} n_j^{(0)} t_i^{(0)} + \frac{M}{R^{(0)}} \phi_x^{(0)} = 0, \quad z = 1, \tag{5.4e}$$
$$w^{(0)} = 0, \quad z = 1, \tag{5.4f}$$
$$\phi_z^{(0)} = 0, \quad z = 1. \tag{5.4g}$$
This eigenvalue problem was examined by Nield (1964).

(b) *The adjoint problem*

We shall need the problem adjoint to system (5.4). We define the adjoint solutions $(\tilde{v}_i, \tilde{\phi}, \tilde{p})$ by the integral relation
$$\langle \tilde{v}_i (\sigma_{ij,j}^{(0)} + R^{(0)} \phi^{(0)} k_i) + \tilde{\phi} (\nabla^2 \phi^{(0)} + R^{(0)} w^{(0)}) \rangle = \langle v_i^{(0)} (\tilde{\sigma}_{ij,j} + R^{(0)} \tilde{\phi} k_i) + \phi^{(0)} (\nabla^2 \tilde{\phi} + R^{(0)} \tilde{w}) \rangle. \tag{5.5}$$
Using the divergence theorem, the continuity equation, boundary conditions (5.4*d*)–(5.4*g*) and the requirement of X_0 periodicity results in the adjoint system
$$\tilde{\sigma}_{ij,j} + R^{(0)} \tilde{\phi} k_i = 0, \tag{5.6a}$$
$$\nabla^2 \tilde{\phi} + R^{(0)} \tilde{w} = 0, \tag{5.6b}$$
$$\tilde{v}_{i,i} = 0, \tag{5.6c}$$
$$\tilde{v}_i = \tilde{\phi} = 0, \quad z = 0, \tag{5.6d}$$
$$\tilde{\sigma}_{ij} n_j^{(0)} t_i^{(0)} = 0, \quad z = 1, \tag{5.6e}$$
$$\tilde{w} = 0, \quad z = 1, \tag{5.6f}$$
$$\tilde{\phi}_z - \frac{M}{R^{(0)}} \tilde{u}_x = 0, \quad z = 1.$$

We now seek to correct Nield's results for small but non-zero C.

(c) *The second-order problem*

At $O(C)$, (5.1*e*) takes the form

$$\eta^{(1)}_{xx} - G\eta^{(1)} = [\sigma^{(0)}_{ij} n^{(0)}_j n^{(0)}_i]_{z=1}, \tag{5.7}$$

from which we can obtain $\eta^{(1)}(x)$. The remaining $O(C)$ system takes the form

$$\sigma^{(1)}_{ij,j} + R^{(0)}\phi^{(1)}k_i = -R^{(1)}\phi^{(0)}k_i, \tag{5.8a}$$

$$\nabla^2\phi^{(1)} + R^{(0)}w^{(1)} = -R^{(1)}w^{(0)}, \tag{5.8b}$$

$$v^{(1)}_{i,i} = 0, \tag{5.8c}$$

$$v^{(1)}_i = \phi^{(1)} = 0, \quad z = 0, \tag{5.8d}$$

$$\sigma^{(1)}_{ij} n^{(0)}_j t^{(0)}_i + \frac{M}{R^{(0)}}\phi^{(1)}_x = \bar{A}_1 + R^{(1)}\bar{B}_1, \tag{5.8e}$$

$$\phi^{(1)}_z = 0, \tag{5.8f}$$

$$w^{(1)} = 0, \tag{5.8g}$$

where

$$\bar{A}_1 = M\eta^{(1)}_x, \tag{5.9a}$$

$$\bar{B}_1 = M\phi^{(0)}_x / R^{(0)2}. \tag{5.9b}$$

There exists a solution of system (5.8) only if the following orthogonality condition with $(\tilde{v}_i, \tilde{\phi})$ holds:

$$\langle \tilde{v}_i(\sigma^{(1)}_{ij,j} + R^{(0)}\phi^{(1)}k_i) + \tilde{\phi}(\nabla^2\phi^{(1)} + R^{(0)}w^{(1)})\rangle = -R^{(1)}\langle \tilde{w}\phi^{(0)} + w^{(0)}\tilde{\phi}\rangle. \tag{5.10}$$

When condition (5.10) is simplified using the divergence theorem and boundary conditions (5.4*d*)–(5.4*g*) and (5.8*d*)–(5.8*g*), we find that

$$R^{(1)}\left\{\langle w^{(0)}\tilde{\phi} + \tilde{w}\phi^{(0)}\rangle + \int_X \bar{B}_1\tilde{u}\right\} = -\int_X \bar{A}_1\tilde{u}. \tag{5.11}$$

As before, both the first-order problem and its adjoint have separable solutions proportional to $\exp\{i\alpha x\}$. From (5.7), $\eta^{(1)} \sim \exp\{i\alpha x\}$ and thus the surface integrals in (5.11) are non-trivial. Thus, using (5.9) and the continuity condition (5.8*c*), we have

$$R^{(1)} = \frac{-M\int_X \eta^{(1)}\tilde{w}_z}{\langle w^{(0)}\tilde{\phi} + \tilde{w}\phi^{(0)}\rangle + \dfrac{M}{R^{(0)2}}\displaystyle\int_X \phi^{(0)}\tilde{w}_z}. \tag{5.12}$$

Once again, using (5.7),

$$\eta^{(1)} = -\frac{1}{\alpha^2 + G}[\sigma^{(0)}_{ij} n^{(0)}_j n^{(0)}_i]_{z=1}$$

and we find that

$$R^{(1)}_L = \frac{M}{\alpha^2 + G}\left\{\int_X \tilde{w}_z \sigma^{(0)}_{ij} n^{(0)}_j n^{(0)}_i\right\}\left\{\langle w^{(0)}\tilde{\phi} + \tilde{w}\phi^{(0)}\rangle + \frac{M}{R^{(0)2}}\int_X \phi^{(0)}\tilde{w}_z\right\}^{-1}. \tag{5.13}$$

The sign of $R^{(1)}_L$ is thus determined by the product on the surface of the normal stress and $\tilde{w}_z$. Equation (5.12) illustrates quite directly that $R^{(1)}_L$ is zero when there is no thermocapillarity, i.e. for $M = 0$. In fact, it is possible to show that $R^{(2)}$, $R^{(3)}$, ... are all zero for an insulated-top free surface for $M = 0$. This is the case because, for $M \equiv 0$, η is determined by (5.1*e*), which decouples from (5.1*f*) and the stationary form of (5.1*g*).

M	$\alpha^{(0)}$	$R_L^{(0)}$	$R_L^{(0)2}$	$R_L^{(1)}$
0	2·10	25·86	669	0
5	2·10	25·10	630	44·9
10	2·08	24·31	591	87·8
15	2·08	23·48	551	126
20	2·06	22·61	511	162
25	2·06	21·69	471	190
30	2·04	20·73	420	215
35	2·04	19·70	388	229
40	2·04	18·61	346	234
45	2·02	17·44	304	232
50	2·02	16·17	261	212
55	2·02	14·78	218	176
60	2·02	13·22	175	120
65	2·02	11·44	131	41·1
70	2·02	9·300	86·5	−49·2
75	2·02	6·455	41·7	−70·0

TABLE 2. Results of numerical calculations for the linear theory with stationary onset of buoyancy–thermocapillary layers with $C \to \infty$. $\alpha^{(0)}$ is the optimal wavenumber. The critical Rayleigh number of linear theory is given by $R_L^{(0)2} + 2R_L^{(0)}R_L^{(1)}C + O(C^2)$.

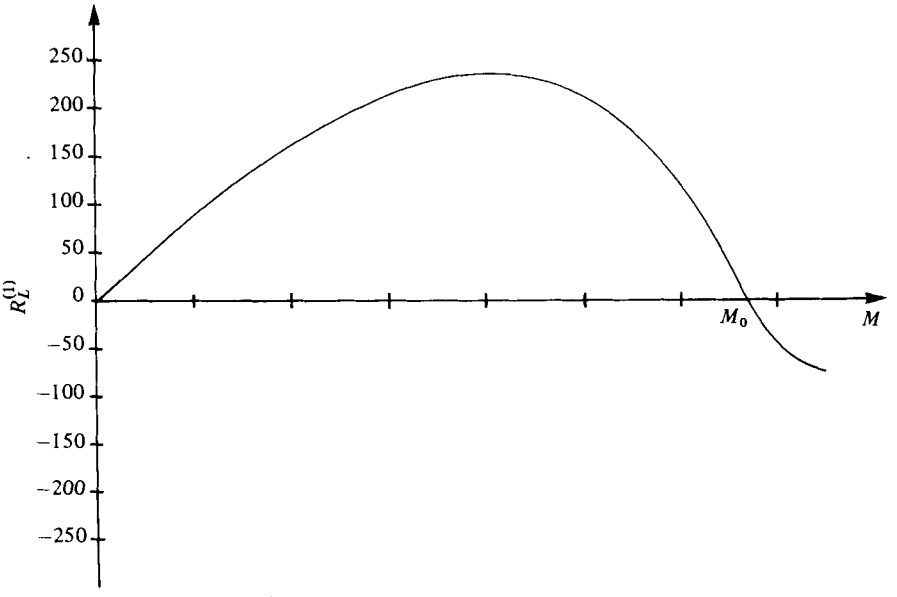

FIGURE 5. The function $R_L^{(1)}(M)$ for the present two-dimensional linear theory of buoyancy–thermocapillary convection in layers. $R_L^{(1)}$ gives the $O(C)$ correction to the number $R_L^{(0)}$. $M_0 \approx 67$ is the value of the Marangoni number for which $R_L^{(1)} = 0$.

Hence, in the linear theory and with the assumptions of an insulated surface and stationary onset, the surface displacement may be calculated exactly by solution of the normal stress balance, using the stress calculated on the fixed domain $0 \leqslant z \leqslant 1$.

We have computed the first-order fields and their adjoints by methods described in the appendix, and have evaluated $R^{(1)}(M)$ using (5.12). The results are given in table 2 and figure 5 for the special case $G = 0$. The general case for $G \neq 0$ can be obtained

through (5.13) by multiplying the above value by $\alpha^2/(\alpha^2+G)$. Clearly, the $G = 0$ case must be handled with caution since the long-wave limit, $\alpha \to 0$, is singular. This limiting case is discussed in detail by Smith (1966).

We can see from figure 5 that, for $0 \leqslant M < M_0 \approx 67$, $R_L^{(1)} > 0$, indicating that weak surface deflexions stabilize the layer; the layer is destablized for $M > M_0$.

The latter result is in accord with the calculations of Scriven & Sternling (1964) and Smith (1966) who found a destabilization with increasing capillary number for pure thermocapillary convection (i.e. $R = 0$). Scriven & Sternling conjectured (incorrectly) that surface deflexions will in general be destabilizing for free-surface instability problems. This conjecture is in accord with one's expectation that an increase in the degree of freedom of a mechanical system will, if anything, lower the stability limits. In the case of convection driven primarily by buoyancy, however, the opposite is the case. The reason is that the normal stress on the surface is negative (positive) above upwelling (downwelling) fluid in pure buoyancy-driven (thermocapillary) convection (see Jeffreys 1951; Davis & Segel 1963; Scriven & Sternling 1964; Kayser & Berg 1973). Application of (5.13), which is valid for small C, leads immediately to the result that surface deflexions for predominantly buoyancy-driven convection are *stabilizing*. A physical interpretation of this result is given in § 6.

6. Discussion

The analysis of §§ 2–4 shows how energy stability theory can be applied to two-dimensional buoyancy–thermocapillary convection in fluid layers with free surfaces. The *nonlinear* Euler–Lagrange system (3.17) governs such a theory in that stability is guaranteed if $R < R_E$, where R_E is the smallest positive eigenvalue of system (3.17). The value of R_E does not depend on the spatial period $2\pi/\alpha$ of the disturbance since R_E is obtained as the minimum over all α. However, R_E does depend on the physical parameters of the problem, namely M, G and C, as well as on the disturbance amplitude ϵ.

In order to assess the implications of the theory for a specific case, we have examined the limiting case of small capillary number C, where $C = \mu\kappa/\sigma_0 d$. Small C is equivalent to a large mean surface tension σ_0, a case for which the free surface is nearly flat. In this case weak surface deflexions act to *increase* R_E, a stabilization that is due to the dynamic effect of the surface corrugations producing normal stresses that counteract the tendency toward instability.

The results of § 5 for the linear theory problem for $C \ll 1$ likewise show that free-surface deflexions can have a stabilizing effect and hence increase the critical value R_L. This occurs when M is non-zero, but sufficiently small. In this case, in which the convection is dominated by buoyancy, the free surface is elevated above a rising current (Jeffreys 1951). This rising fluid is warm but is cooler than the equivalent fluid in the case of a flat interface, since the bulge allows augmented cooling. Similarly, at a descending current, i.e. a depressed surface, the temperature is higher than that in the flat interface case. Hence, the effect of the surface corrugation is to *decrease* the surface temperature gradient and hence the surface-tension gradient compared to the flat-interface case. This leads to a thermocapillary stress which is weaker than its value for a flat interface, leading to a stabilization.

The *a priori* knowledge that $R_E \leqslant R_L$ limits the region of validity of the small-C results. For example at $M = 45$ the numerical results of tables 1 and 2 indicate that

$$R_E \doteq 13{\cdot}6 + 1320C + O(C^2) \tag{6.1a}$$

and

$$R_L \doteq 17{\cdot}44 + 232C + O(C^2). \tag{6.1b}$$

These results are limited by the condition $R_E \leqslant R_L$. However, a more stringent limitation of the simultaneous validity of forms (6.1) is that $1320C/13{\cdot}6 \ll 1$ or $C \ll 10^{-2}$, a condition that includes many physically realistic cases but fails well before $R_E \approx R_L$. The limitation is, of course, only on the small-C expansion of the Euler–Lagrange system, not of the energy stability theory in general.

Results such as those of figure 4 are inherently ones of conditional stability. The value of R_E depends in general on ϵ. However, in our example the leading term $\eta^{(0)}$, which is the solution of a nonlinear problem (4.3*a*), fortuitously is ϵ independent, $\eta^{(0)} \equiv 0$. Hence, for our example, $C \to 0$ is equivalent to a small-ϵ solution. In the small-C case the mean surface tension is too large to allow sharply peaked two-dimensional interfacial shapes, such as that shown in figure 1, to be generated. Furthermore, we find no strong tendency toward instability even if initially such a spike were present. This is because we have treated the *two-dimensional problem* in which a spike is really a thin sheet. The three-dimensional version of the present theory might well exhibit the effects of the tendency toward droplet break-off as illustrated in figure 1.

7. Generalization

In the formulation of the power integrals in § 3, we considered evolution equations that are symbolically of the type

$$P^{-1}\,\partial v_i/\partial t = F_i \tag{7.1a}$$

and

$$\partial\theta/\partial t = G^{(1)}. \tag{7.1b}$$

As is usual, we then multiplied (7.1*a*) by v_i and (7.1*b*) by θ to form the power integrals

$$\frac{d}{dt}\langle \tfrac{1}{2}P^{-1}v_i v_i\rangle = \langle v_i F_i\rangle, \tag{7.2}$$

$$\frac{d}{dt}\langle \tfrac{1}{2}\theta^2\rangle = \langle \theta G^{(1)}\rangle. \tag{7.3}$$

The crucial step in the free-surface analysis was the observation that the kinematic boundary condition is also an evolution equation that is symbolically of the form

$$\partial\eta/\partial t = S^{(1)}. \tag{7.4}$$

We then formed a third power integral,

$$\frac{d}{dt}\int_{S(t)} \tfrac{1}{2}W\eta^2/N = \int_{S(t)} \tfrac{1}{2}WS^{(1)}/N, \tag{7.5}$$

where W is a known weight function. (We actually used $W \equiv 1$.) The energy theory follows by combining (7.2), (7.3) and (7.5) into a simple generalized energy functional

and formulating a variational principle. In effect, we have considered the governing equation as a vector system for the vector

$$\boldsymbol{\psi} = \begin{Bmatrix} v_i \\ \theta_i \\ \eta \end{Bmatrix}$$

and considered the set of evolution equations of the type

$$\psi_t = F.$$

The generalized energy is obtained by forming the scalar product

$$\|\boldsymbol{\psi}\|^2 = \boldsymbol{\psi} . \begin{Bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/N \end{Bmatrix} \boldsymbol{\psi} = \langle v_i v_i + \lambda \theta^2 \rangle + \int_{S(t)} \eta^2 / N,$$

where $\lambda > 0$. We find that

$$\frac{d}{dt} \|\boldsymbol{\psi}\|^2 = G, \tag{7.6}$$

and the energy theory follows from form (7.6).

It is worth noting that this formulation reduces to the classical one for planar, stress-free surfaces (Davis 1969) in the limit $\eta \to 0$. That is, by treating the kinematic condition as an evolution equation, analyzing as described and then letting $\eta \to 0$, we regain the kinematic condition $w = 0$ on $z = 1$ as a *natural boundary condition.*

If the basic state were dynamic, involving a basic shear flow, precisely the *same formalism* would be valid although new terms would appear in the functional G.

Clearly, the *same procedure* will go through if three-dimensional disturbances are allowed. Instead of having derivatives ∂/∂_s with respect to arc length s, $\partial/\partial_s = (1/N)\partial/\partial x$, we would have two independent directional-derivatives $\mathbf{t}^{(\alpha)}.\nabla$, $\alpha = 1, 2$, corresponding to the two independent surface co-ordinates that describe the surface. Rather than using integration by parts, surface divergence theorems are required (Moeckel 1975). In summary then, although the algebraic and geometric complexity is increased, the three-dimensional energy-stability theory in principle proceeds in the same way as does the two-dimensional theory.

An attractive aspect of our formulation is that all bulk-stress effects are given in terms of the stress tensor σ_{ij}. Thus, the equations of motion and the boundary conditions are formulated so as to be valid even when the bulk is a *non-Newtonian fluid.* The same formalism holds.

We now consider the possibility of including more exotic surface properties in the theory. We assume that the interface possesses a surface excess mass density γ_s, a surface tension σ with surface-tension gradient $\nabla_s \sigma$ and surface shear and dilational viscosities μ_s and μ_d respectively. These surface quantities satisfy (Moeckel 1975) balance laws of mass and momentum of the form

$$\frac{\partial \gamma_s}{\partial t} = S^{(2)} \quad \text{on} \quad S(t) \tag{7.7a}$$

and

$$\gamma_s \frac{\partial v_i}{\partial t} = S_i^{(3)} \quad \text{on} \quad S(t), \tag{7.7b}$$

where

$$S^{(2)} = -\nabla_s . (\gamma_s \mathbf{v}) + 2H\gamma_s V \tag{7.7c}$$

and $$S^{(3)} = -\nabla_s.(\gamma_s \mathbf{vv}) + \nabla_s\sigma + \mu_s\nabla_s^2\mathbf{v} + \mu_d\nabla_s(\nabla_s.\mathbf{v}) + \boldsymbol{\sigma}.\mathbf{n} + 2H\sigma V\mathbf{n}. \tag{7.7d}$$

Here $\mathbf{n}$ is the unit outward normal vector to $S(t)$ and H is the mean curvature of $S(t)$. We have assumed that $S(t)$ is a material surface and that the passive gas boundary $S(t)$ exerts negligible forces on $S(t)$. (These restrictions are easily relaxed.) The interface moves with velocity $V\mathbf{n}$.

The forms (7.6a, b), since they involve local time derivatives, should be treated as evolution equations. The power integrals are obtained by respectively multiplying by γ_s and v_i. The results are of the form

$$\frac{d}{dt}\int_{S(t)} \tfrac{1}{2}\gamma_s^2 = \int_{S(t)} \gamma_s S^{(2)} \tag{7.8}$$

and

$$\frac{d}{dt}\int_{S(t)} \tfrac{1}{2}\gamma_s v_i v_i = \int_{S(t)} v_i S_i^{(0)}. \tag{7.9}$$

We then form a generalized energy function E,

$$E = \langle \tfrac{1}{2}P^{-1}v_i v_i\rangle + \lambda\langle \tfrac{1}{2}\theta^2\rangle + \int_{S(t)} \{\tfrac{1}{2}\eta^2/N + \tfrac{1}{2}\gamma_s^2 + \tfrac{1}{2}\gamma_s v_i v_i\}, \tag{7.10}$$

where $\lambda > 0$ is the linking parameter. The statement of this functional is again the crux of the theory since the formulation of the variational problem is then straightforward.

If we imagine taking the limit of $\gamma_s, \mu_s, \mu_d \to 0$, then the Euler–Lagrange equations emerging from form (7.10) *automatically* give the same system as obtained by letting $\gamma_s, \mu_s, \mu_d = 0$ *a priori* and applying the stress jumps (2.11d) as side conditions. Again, our procedures yield the appropriate generalization.

Hence, the recipe for the full formulation involves recognizing that a vector ψ must be defined. To obtain (7.10), we take

$$\psi = \begin{Bmatrix} v_i \\ \theta \\ \eta \\ \gamma_s \\ [v_i]_{S(t)} \end{Bmatrix}. \tag{7.11}$$

The elements of ψ consist of all those variables that explicitly evolve in time, i.e. those upon which the local time derivative is applied. The full system is

$$\psi_t = F \tag{7.12}$$

and

$$\frac{d}{dt}\|\psi\|^2 = G. \tag{7.13}$$

All of the appropriate surface boundary conditions of the Euler–Lagrange system for equation (7.13) are *natural boundary conditions*. Energy stability results can then be obtained either analytically or numerically.

In all the above discussions, it has been presumed that the interface $S(t)$ does not intersect a solid boundary. Such an intersection is called a contact (or common or three-phase) line. When the no-slip condition is applied at the solid wall, any movement of the contact line (due to motion in the basic state or due to disturbances) is

associated with a non-integrable singularity (Dussan V. & Davis 1974). With no modification of the model, the power integrals analogous to those in § 3 would not exist. However, effective slip (see Dussan V. 1979) inserted near the contact line relieves this problem and hence makes our energy theory applied to such problems well defined.

S. H. D. and G. M. H. are grateful for the partial support of this work through grants of the National Science Foundation, Fluid Mechanics Program, and United States Department of Energy, respectively.

Appendix

We record here the expansions of the natural boundary conditions (3.17*e, f, g, h*) to arrive at (4.11), (4.12*a–c*). We begin by writing (3.17*e*) in its normal and tangential components:

$$\tfrac{1}{2}K = C[\sigma_{ij}n_in_j + \tfrac{1}{2}MK(\phi/R\lambda - \eta)] \quad \text{on} \quad z = 1+\eta, \tag{A 1}$$

$$\sigma_{ij}n_jt_i = -\tfrac{1}{2}M[\phi'/R\lambda - \eta_x]/N \quad \text{on} \quad z = 1+\eta. \tag{A 2}$$

By expanding all variables in powers of C, we get from (A 1) at $O(1)$

$$K^{(0)} = 0, \tag{A 3a}$$

and at $O(C)$

$$K^{(1)} = \eta^{(1)}_{xx} = 2\sigma^{(0)}_{ij}n^{(0)}_in^{(0)}_j. \tag{A 3b}$$

Equations (A 3*a, b*) are equations (4.3*a*) and (4.11) of the text. In what follows, we use the solution to (A 3*a*), namely $\eta^{(0)} \equiv 0$, and the expansions for N, n_i, t_i given in (4.5).

By expanding (A 2) we get at $O(1)$

$$\sigma^{(0)}_{ij}n^{(0)}_jt^{(0)}_i + \frac{1}{2}\frac{M\phi^{(0)}_x}{R^{(0)}\lambda^{(0)}} = 0, \quad z = 1, \tag{A 4a}$$

and

$$\sigma^{(1)}_{ij}n^{(0)}_it^{(0)}_i + \frac{1}{2}\frac{M\phi^{(1)}_x}{R^{(0)}\lambda^{(0)}} = -[\sigma^{(0)}_{ij}(n^{(1)}_jt^{(0)}_i + n^{(0)}_jt^{(1)}_i) + \sigma^{(0)}_{ij,z}n^{(0)}_jt^{(0)}_i\eta^{(1)}]$$
$$+ \frac{M}{2R^{(0)}\lambda^{(0)}}\left[\phi^{(0)}_x\left(\frac{\lambda^{(1)}}{\lambda^{(0)}} + \frac{R^{(1)}}{R^{(0)}}\right) - \phi^{(0)}_{xz}\eta^{(1)} + \eta^{(1)}_x\phi^{(0)}_z\right] + \tfrac{1}{2}M\eta^{(1)}_x, \quad z = 1. \tag{A 4b}$$

Equation (A 4*a*) corresponds to (4.7*a*) and (A 4*b*) to (4.12*a*) with the definitions

$$A_1 = \tfrac{1}{2}M\eta^{(1)}_x - [\sigma^{(0)}_{ij}(n^{(1)}_jt^{(0)}_i + n^{(0)}_jt^{(1)}_i) + \sigma^{(0)}_{ij,z}n^{(0)}_jt^{(0)}_i\eta^{(1)}]$$
$$+ \frac{M}{2R^{(0)}\lambda^{(0)}}\left[\phi^{(0)}_x\frac{\lambda^{(1)}}{\lambda^{(0)}} - \phi^{(0)}_{xz}\eta^{(1)} + \eta^{(1)}_x\phi^{(0)}_z\right], \tag{A 5}$$

$$B_1 = \frac{M\phi^{(0)}_x}{2\lambda^{(0)}R^{(0)2}}. \tag{A 6}$$

Equation (3.17*f*) is

$$\phi_{,i}n_i - \frac{1}{2}\frac{M}{R\lambda}v'_it_i/N - \lambda R(1-N) = 0, \tag{A 7}$$

where we have used the result $N = 1 + O(C^2)$. Thus to $O(1)$ this becomes

$$\phi^{(0)}_z - \frac{Mu^{(0)}_x}{2R^{(0)}\lambda^{(0)}} = 0, \quad z = 1, \tag{A 8}$$

and at $O(C)$

$$\phi_z^{(1)} - \frac{Mu_x^{(1)}}{2R^{(0)}\lambda^{(0)}} = -[\phi_z^{(0)}\eta^{(1)} - \phi_x^{(0)}\eta_x^{(1)}] + \frac{M}{2R^{(0)}\lambda^{(0)}}\left[u_{xz}^{(0)}\eta^{(1)} + u_z^{(0)}\eta_x^{(1)} + w_x^{(0)}\eta_x^{(1)} - \left(\frac{R^{(1)}}{R^{(0)}} + \frac{\lambda^{(1)}}{\lambda^{(0)}}\right)u_x^{(0)}\right], \quad z = 1. \quad \text{(A 9)}$$

Equation (A 8) is equation (4.7b) in the text and by comparing (A 9) and (4.12b) we find

$$C_1 = [\phi_x^{(0)}\eta_x^{(1)} - \phi_{zz}^{(0)}\eta^{(1)}] + \frac{M}{2R^{(0)}\lambda^{(0)}}\left[u_{xz}^{(0)}\eta^{(1)} + u_z^{(0)}\eta_x^{(1)} + w_x^{(0)}\eta_x^{(1)} - \frac{\lambda^{(1)}}{\lambda^{(0)}}u_x^{(0)}\right], \quad \text{(A 10)}$$

$$D_1 = -\frac{Mu_x^{(0)}}{2\lambda^{(0)}R^{(0)2}}. \quad \text{(A 11)}$$

It remains to expand (3.17g). It is best to begin by noting that

$$((\sigma v_i n_i N)' N^{-3})' = w_{xx}^{(0)} + C\left[w_{xx}^{(1)} + w_{xxz}^{(0)}\eta^{(1)} - (u^{(0)}\eta_x^{(1)})_{xx} + (w_z^{(0)}\eta_x^{(1)})_x + w_{xz}^{(0)}\eta_x^{(1)} - \frac{M}{R^{(0)}\lambda^{(0)}}(w^{(0)}\phi^{(0)})_{xx}\right] + O(C^2), \quad z = 1. \quad \text{(A 12)}$$

Therefore, equation (3.17g) becomes, at $O(1)$,

$$w_{xx}^{(0)} = 0, \quad z = 1, \quad \text{(A 13)}$$

and at $O(C)$, using many previous results,

$$w_{xx}^{(1)} \equiv \hat{E}_1 = (u^{(0)}\eta_x^{(1)})_{xx} - (w_z^{(0)}\eta_x^{(1)})_x - w_{xz}^{(0)}\eta_x^{(1)} + \frac{M}{R^{(0)}\lambda^{(0)}}(w^{(0)}\phi^{(0)})_{xx} - (u^{(0)}\eta_{xx}^{(1)})_{xx} - w_{xxz}^{(0)}\eta^{(1)} + \tfrac{1}{2}\epsilon_{ij}^{(0)}\epsilon_{ij}^{(0)} + \phi_i^{(0)}\phi_{,i}^{(0)} - \beta^{(0)} + M\left[\frac{u_z^{(0)}\phi_x^{(0)}}{R^{(0)}\lambda^{(0)}} + u_x^{(0)}\right], \quad z = 1. \quad \text{(A 14)}$$

Thus by referring to (4.12), we have

$$E_1 = \int^x \int^x \hat{E}_1,$$

or in terms of normal modes

$$E_1 = M\int^x [u^{(0)}]_{z=1} + \text{terms proportional to } \exp\{2i\alpha x\}.$$

In order to satisfy constraint (3.17h), we choose $\beta^{(0)} = 0$.

We now wish to briefly describe the numerical method used to compute the results of figures 4 and 5. In order to apply (4.16) and (5.3) for $R_E^{(1)}$ and $R_L^{(1)}$ respectively, it is necessary to solve for the zero-order eigenfunctions of systems (4.2), (4.6), (4.7) and (5.4). We choose to do this in terms of the exact solution to the reduced sixth-order problem which is equivalent to (4.2a, b, c) and (5.4a, b, c) (Pellew & Southwell 1940). Eliminating $u^{(0)}$, $v^{(0)}$, $\phi^{(0)}$, $p^{(0)}$ from each of these, we find in the usual way that the function $w_0(z)$, where $w^{(0)}(x, z) = w_0(z)\, e^{i\alpha x}$, satisfies

$$(D^2 - \alpha^2)^3 w_0 = -\mathscr{R}^2\alpha^2 w_0 \quad \text{(A 15)}$$

with

$$w_0 = Dw_0 = (D^2 - \alpha^2)^2 w_0 = 0, \quad z = 0, \quad \text{(A 16}a\text{)}$$

$$w_0 = D^2 w_0 + \frac{M}{2\lambda R\mathscr{R}}(D^2 - \alpha^2)^2 w_0 = D(D^2 - \alpha^2)^2 w_0 + \frac{\mathscr{R}\alpha^2 M}{2\lambda R} Dw_0 = 0, \quad z = 1, \quad \text{(A 16}b\text{)}$$

in the case of *energy theory*, and

$$w_0 = Dw_0 = (D^2-\alpha^2)^2 w_0 = 0, \quad z = 0, \tag{A 17a}$$

$$w_0 = D^2 w_0 + \frac{M}{R^2}(D^2-\alpha^2)^2 w_0 = D(D^2-\alpha^2)^2 w_0 = 0, \quad z = 1, \tag{A 17b}$$

in the case of *linear theory*. Here $D = d/dz$, and for simplicity we have written $(\lambda^{(0)}, R^{(0)}, \mathscr{R}^{(0)})$ as $(\lambda, R, \mathscr{R})$. (For the linear theory, $\lambda = 1, \mathscr{R} = R$.) The exact solution of (A 15) can be obtained in terms of complex exponential functions, $w_0 = \sum_{n=1}^{6} A_n \exp\{q_n z\}$, as shown by Pellew & Southwell (1940). At an eigenvalue, the 6×6 coefficient determinant corresponding to (A 16) or (A 17) vanishes. The eigenvalues $R_E^{(0)}$, $R_L^{(0)}$ were determined by iteration and the corresponding constants $\{A_n\}$ by Gaussian elimination. All other field variables, $\phi^{(0)}$, $p^{(0)}$, $\sigma_{ij}^{(0)} n_i^{(0)} n_j^{(0)}$ may be determined, in terms of normal mode amplitudes ϕ_0, p_0, $\sigma_{0ij} n_i n_j$, directly from the $\{A_n\}$. The solution to the problem adjoint to the linear stability system was determined by identical means. Simple multiplications and an integration over z gives the numerical results for $R_E^{(1)}$ and $R_L^{(1)}$ as

$$R_E^{(1)} = \frac{\dfrac{2M}{\alpha^2}[Dw_0(-p_0+2Dw_0)]_{z=1}}{\left(\lambda^{(0)}+\dfrac{1}{\lambda^{(0)}}\right)\displaystyle\int_0^1 w_0\phi_0\,dz + \frac{M}{\lambda^{(0)}R^{(0)}}[\phi_0 Dw_0]_{z=1}}, \tag{A 18}$$

$$R_L^{(1)} = \frac{\dfrac{M}{\alpha^2+G}[D\tilde{w}(-p_0+2Dw_0]_{z=1}}{\displaystyle\int_0^1 (\tilde{w}\phi_0 + w_0\tilde{\phi}) + \frac{M}{R^{(0)2}}[\phi_0 D\tilde{w}]_{z=1}}. \tag{A 19}$$

The values of $R_L^{(1)}$ and $R_E^{(1)}$ were found to be sensitive to $\alpha^{(0)}$ and $\lambda^{(0)}$. For example, at $M = 40$, an inaccuracy of 3 % in $\lambda^{(0)}$ causes an error of only 0·05 % in $R^{(0)}$, but a 4 % error in $R^{(1)}$. It was necessary to determine $\alpha^{(0)}$ to within 0·5 % and $\lambda^{(0)}$ to within 1 % to achieve reasonable accuracy.

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